14 Separation of Variables Method

Consider, for example, the Dirichlet problem

$$u_t = Du_{xx} \qquad 0 < x < l\ ,\ t > 0$$

$$u(x,0) = f(x) \qquad 0 < x < l$$

$$u(0,t) = 0 = u(l,t) \qquad t > 0$$

Let $u(x,t) = T(t)\phi(x)$; now substitute into the equation:

$$\frac{dT}{dt}\phi = DT\frac{d^2\phi}{dx^2} \quad \text{or}$$

$$\frac{1}{DT}\frac{dT}{dt} = \frac{1}{\phi}\frac{d^2\phi}{dx^2} \quad .$$

But the left-hand side depends only on the (independent) variable t, while the right-hand side depends only on x, so this expression must be constant:

$$\frac{1}{DT}\frac{dT}{dt} = \frac{1}{\phi}\frac{d^2\phi}{dx^2} = -\lambda$$

Remark: I will get back to why the negative sign is used on the right side shortly; it is done for convenience because the constant turns out to be a negative real number, making λ easily remembered as being positive.

For the T equation, $dT/dt = -\lambda DT$, the rate of change of T is proportional to T, so $T(t)$ must be an exponential function; namely, up to a multiplication constant, $T(t) = e^{-\lambda Dt}$. Recall from an earlier discussion that for diffusion we expect dissipation of the features of $f(x)$ over time, so it is reasonable to have $T(t) \to 0$ as $t \to \infty$. For this to happen, we would expect $\lambda \geq 0$.

Exercise: Obtain the $T(t)$ by method of integrating factors.

Applying our boundary conditions to $u = T\phi$, we have, for example, $0 = u(0,t) = T(t)\phi(0) \rightarrow \phi(0) = 0$. So the spatial problem is

$$\begin{aligned} &\tfrac{d^2\phi}{dx^2} + \lambda\phi = 0 \qquad 0 < x < l \\ &\phi(0) = 0 = \phi(l) \end{aligned} \tag{1}$$

But we do not know the value of the constant λ in (1), and since we must ultimately satisfy the non-zero initial condition when we return to the u problem, $\phi(x)$ can not be the zero function. Hence (1) is an **eigenvalue problem (EVP)**. So we want all solutions $\{\lambda, \phi(\cdot)\}$ such that ϕ is a non-trivial function. Put another way, we seek all (real) constants λ that give us a non-zero $\phi(x)$, up to a multiplicative constant.

Solving the EVP

First suppose we have a solution pair $\{\lambda, \phi\}$ to (1). Multiply the equation for ϕ by ϕ and integrate:

$$\int_0^l \phi \frac{d^2\phi}{dx^2} dx + \lambda \int_0^l \phi^2 dx = 0$$

By integration-by-parts

$$\int_0^l \phi \frac{d^2\phi}{dx^2} dx = \phi \frac{d\phi}{dx}\Big|_0^l - \int_0^l \left(\frac{d\phi}{dx}\right)^2 dx \quad .$$

Hence

$$-\int_0^l \left(\frac{d\phi}{dx}\right)^2 dx + \lambda \int_0^l \phi^2 dx = 0 \quad \text{or} \quad \lambda = \int_0^l \left(\frac{d\phi}{dx}\right)^2 dx \Big/ \int_0^l \phi^2 dx \geq 0 \quad .$$

Again, by our sign convention, this shows $\lambda \geq 0$ (assuming λ is a real constant; later in the Notes we will show λ must be real). The ratio of integrals describing λ is called the **Rayleigh Quotient**. We will return to discuss it later in the course when we discuss more general eigenvalue problems.

Now if $\lambda = 0$ in (1), then $\phi(x) = Ax + B$, but applying the b.c.s gives $\phi(x) \equiv 0$. Therefore, $\lambda = 0$ is not an eigenvalue. For $\lambda > 0$, by the *characteristic equation method*, since the equation in (1) is a constant coefficient equation, $\phi(x) = e^{rx} \rightarrow e^{rx}r^2 + \lambda e^{rx} = 0 \ \rightarrow r^2 = -\lambda$, which implies

$r = \pm i\sqrt{\lambda}$. While $e^{i\sqrt{\lambda}}, e^{-i\sqrt{\lambda}}$ is a valid fundamental set of solutions, we want to deal with real-valued functions. Hence, an equivalent fundamental set of solutions we use is $\cos(\sqrt{\lambda}x), \sin(\sqrt{\lambda}x)$, so the general solution is $\phi(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x)$. Apply the b.c.s: $\phi(0) = 0 = A$ and $\phi(l) = 0 = B\sin(\sqrt{\lambda}l)$. But $B \neq 0$, so $\sin(\sqrt{\lambda}l) = 0 \to \sqrt{\lambda}l = n\pi$, for $n = 1, 2, 3, \ldots$. That is, for each positive integer n ($n \in \mathbb{N}$), $\lambda = \lambda_n = (\frac{n\pi}{l})^2$ is an eigenvalue for the problem, and $\sin(\frac{n\pi x}{l})$ is an eigenfunction associated with λ_n, so call it $\phi_n(x)$. So we have an infinite number of eigenvalue-eigenfunction pairs $\{\lambda_n, \phi_n(x)\} = \{(\frac{n\pi}{l})^2, \sin(\frac{n\pi x}{l})\}_{n=1}^{\infty}$, and since $T(t)$ also depends on λ_n, write $T_n(t) = b_n e^{-n^2\pi^2 Dt/l^2}$. Thus, $u(x,t) = u_n(x,t) = T_n(t)\phi_n(x) = b_n e^{-n^2\pi^2 Dt/l^2}\sin(\frac{n\pi x}{l})$ satisfies the pde and boundary conditions for each $n \in \mathbb{N}$.

What about the initial condition? Since any finite linear combination of such u_n 's is also a solution to the pde and b.c.s, then if $f(x)$ is a sum of u_n 's, we just *match coefficients*. For example, if $f(x) = 3\sin(\frac{9\pi x}{l}) - 5\sin(\frac{15\pi x}{l})$, then $u(x,t) = 3e^{-81\pi^2 Dt/l^2}\sin(\frac{9\pi x}{l}) - 5e^{-225\pi^2 Dt/l^2}\sin(\frac{15\pi x}{l})$. But most initial conditions $f(x)$ are **not** a linear combination of eigenfunctions. For an arbitrary continuous function on $0 \leq x \leq l$, we may need all possible contributions u_n so write

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 Dt/l^2}\sin(\frac{n\pi x}{l}) \quad .$$

To satisfy the initial condition means we need

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(\frac{n\pi x}{l}) \quad . \tag{2}$$

But what does this mean? By combining all possible contributions in an infinite series we are employing an *extended* superposition principle, and in (2), the series represents the **Fourier sine series for** f if the $b_n's$ are the appropriate Fourier coefficients. To obtain these, multiply (2) by some arbitrary eigenfunction $\sin(\frac{m\pi x}{l})$ and integrate:

$$\int_0^l f(x)\sin(\frac{m\pi x}{l})dx = \int_0^l \sum_{n=1}^{\infty} b_n \sin(\frac{n\pi x}{l})\sin(\frac{m\pi x}{l})dx$$

$$= \sum_{n\geq 1} b_n \int_0^l \sin(\frac{n\pi x}{l})\sin(\frac{m\pi x}{l})dx \ .$$

We will leave justification for interchanging integration and infinite summation for later. By the trig addition formulas,

$$\sin(A)\sin(B) = \begin{cases} \sin^2(A) = \frac{1}{2} - \frac{1}{2}\cos(2A) & \text{if } B = A \\ \\ \frac{1}{2}[\cos(A-B) - \cos(A+B)] & \text{if } B \neq A \end{cases}$$

so with $A = n\pi x/l$, $B = m\pi x/l$,

$$\int_0^l \sin(\frac{m\pi x}{l})\sin(\frac{m\pi x}{l})dx = \begin{cases} 0 & \text{if } n \neq m \\ \\ \frac{l}{2} & \text{if } n = m \end{cases} \tag{3}$$

Exercise: Work through the details of this calculation yourself.

Formula (3) states that the sequence of eigenfunctions $\{\sin(\frac{n\pi x}{l})\}$ on $(0, l)$ is an **orthogonal sequence**. Thus

$$\int_0^l f(x)\sin(\frac{m\pi x}{l})dx = b_m \frac{l}{2} \ ,$$

hence,

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 Dt/l^2} \sin(\frac{n\pi x}{l})$$

where, for each n,

$$b_n = \frac{2}{l}\int_0^l f(x)\sin(\frac{n\pi x}{l})dx \ \ .$$

There are steps that need justification, but for now consider some specific problems to get the process down.

Example 1:

$$\begin{cases} u_t = Du_{xx} & 0 < x < 1, \ t > 0 \\ u(x,0) = R & 0 < x < 1 \\ u(0,t) = 0 = u(1,t) & t > 0 \end{cases}$$

Here $R =$ constant > 0, and $l = 1$, so we have the above expression for u except now

$$f(x) \equiv R = \sum_{n=1}^{\infty} b_n \sin(n\pi x) \ \ .$$

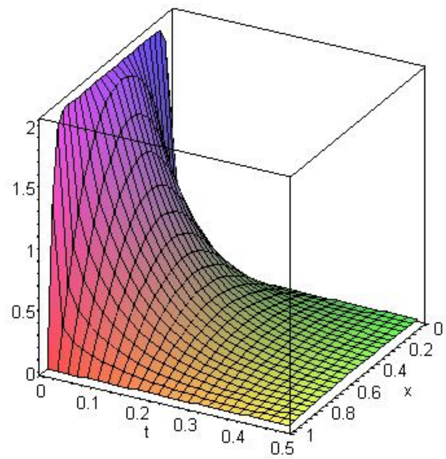

Figure 1: solution to example 1 with $D = 1, R = 2$. Note the rapid decay in the solution, even at the center of the rod.

Now, from the formula for the Fourier coefficients,

$$\int_0^1 R\sin(n\pi x)dx = -\frac{R}{n\pi}\cos(n\pi x)|_{x=0}^{x=1} = \frac{R}{n\pi}\{1-\cos(n\pi)\}$$

$$= \frac{R}{n\pi}\{1-(-1)^n\} = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{2R}{n\pi} & \text{if } n \text{ is odd} \end{cases} .$$

Therefore, the solution is

$$u(x,t) = \frac{4R}{\pi}\sum_{n=1,3,5,\ldots}\frac{e^{-n^2\pi^2 Dt}}{n}\sin(n\pi x) = \frac{4R}{\pi}\sum_{k=1}^{\infty}\frac{e^{-(2k-1)^2\pi^2 Dt}}{2k-1}\sin((2k-1)\pi x) .$$

See figure 1 for a graph of the solution in a specific case.

Example 2:

$$\begin{cases} u_t = u_{xx} & 0 < x < 2\ ,\ t > 0 \\ u(x,0) = x & 0 < x < 2 \\ u(0,t) = 0 = u(2,t) & t > 0\ . \end{cases}$$

Now

$$b_n = \int_0^2 x \sin(\frac{n\pi x}{2})dx = -\frac{2x}{n\pi}\cos(\frac{n\pi x}{2})|_0^2 + \frac{2}{n\pi}\int_0^2 \cos(\frac{n\pi x}{2})dx$$

$$= -\frac{4}{n\pi}\cos(n\pi) = \frac{4}{n\pi}(-1)^{n+1} \quad .$$

Therefore

$$u(x,t) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}e^{-n^2\pi^2 t/4}}{n}\sin(\frac{n\pi x}{2}) \quad .$$

Example 3: Consider the Neumann problem

$$\begin{cases} u_t = u_{xx} & 0 < x < 2\ ,\ t > 0 \\ u(x,0) = x & 0 < x < 2 \\ u_x(0,t) = 0 = u_x(2,t) & t > 0\ . \end{cases}$$

With a change in boundary conditions we must again look at the EVP. With $u = T(t)\phi(x)$, we have

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0 \qquad 0 < x < 2$$

$$\frac{d\phi}{dx}(0) = 0 = \frac{d\phi}{dx}(2)$$

Now if $\lambda = 0$, $\frac{d^2\phi}{dx^2} = 0$, so $\phi(x) = Ax + B$, but the boundary conditions force $A = 0$, so $\phi = B =$ arbitrary constant. Hence, $\lambda = 0$ is an eigenvalue. We can let $\phi \equiv 1$ be the associated eigenfunction. For $\lambda > 0$ we have $\phi(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x)$. For the first b.c., $\frac{d\phi}{dx}(0) = \sqrt{\lambda}B = 0$. Since $\lambda \neq 0$, then $B = 0$. Applying the second b.c. gives $\frac{d\phi}{dx}(2) = -\sqrt{\lambda}A\sin(2\sqrt{\lambda}) = 0$. We can not have $A = 0$, which means we need $\sin(2\sqrt{\lambda}) = 0$. This implies that $2\sqrt{\lambda} = n\pi$, again for $n \in \mathbb{N}$. Finally, $\lambda = \lambda_n = (\frac{n\pi}{2})^2$, with eigenfunctions $\phi_n(x) = \cos(\frac{n\pi x}{2})$, $n = 1, 2, 3, \ldots$. Since the equation did not change, we have $T(t) =$ constant if $\lambda = 0$, otherwise $T_n(t) = e^{-\lambda_n t} = e^{-n^2\pi^2 t/4}$. Employing the superposition principle (writing a linear combination for all the potential contributions), we have

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-n^2\pi^2 t/4}\cos(\frac{n\pi x}{2}) \tag{4}$$

Every term satisfies the pde and b.c.s, and so does the infinite series (4) if we can interchange summation and integration. Letting $t \to 0$, we need

$$x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi x}{2})$$

That is, we want the **Fourier cosine series** for $f(x) = x$ on the interval $(0, 2)$. We play the same game we did in the last examples, namely multiply both sides of the Fourier series by an arbitrary eigenfunction, $\cos(\frac{m\pi x}{2})$, and integrate both sides.

$$\int_0^2 x \cos(\frac{m\pi x}{2})dx = \int_0^2 \frac{a_0}{2} \cos(\frac{m\pi x}{2})dx + \sum_{n\geq 1} a_n \int_0^2 \cos(\frac{n\pi x}{2}) \cos(\frac{m\pi x}{2})dx$$

For $m > 0$ the first integral on the right side is zero. Again by using the trig additions formulas, the eigenfunctions $\{\cos(\frac{n\pi x}{l})\}_{n\geq 0}$ form an orthogonal set of functions on $(0, l)$, where $l = 2$, with

$$\int_0^l \cos(\frac{m\pi x}{2}) \cos(\frac{n\pi x}{2})dx = \begin{cases} l/2 & \text{if } n = m \\ \\ 0 & \text{if } n \neq m \end{cases} \tag{5}$$

Note that we can include $m = 0$ here. That is, to find a_0, we multiply by 1 (The eigenfunction for the zero eigenvalue) and integrate. All the terms in the sum integrate to zero. So the right-hand side becomes just a_0. Otherwise, using $\cos(\frac{m\pi x}{2})$, $m \geq 1$, the right-hand side becomes $a_m \frac{l}{2} = a_m$. For the left-hand side,

$$\int_0^2 x \cos(\frac{m\pi x}{2})dx = \frac{4}{m^2\pi^2} \cos(\frac{m\pi x}{2})|_0^2 = \frac{4}{m^2\pi^2}\{(-1)^m - 1\} \ .$$

So, for $m \geq 1$

$$a_m = \begin{cases} 0 & m = \text{even} \\ \\ -\frac{8}{m^2\pi^2} & m = \text{odd} \end{cases}$$

These are the Fourier cosine coefficients for $f(x) = x$ we were looking for. Putting this into the expression for u gives

$$u(x,t) = 1 - \frac{8}{\pi^2} \sum_{m=1,3,5,\ldots} \frac{e^{-m^2\pi^2 t/4}}{m^2} \cos(\frac{m\pi x}{2})$$

$$= 1 - \frac{8}{\pi^2}\sum_{k=1}^{\infty}\frac{e^{-(2k-1)^2\pi^2 t/4}}{(2k-1)^2}\cos(\frac{(2k-1)\pi x}{2}) \ .$$

Remark: In the series (4) it is classical to write the first term as $a_0/2$ rather than a_0 because it simplifies slightly the description of the Fourier coefficients. Due to the fact we have a Neumann boundary condition on **both** ends of the interval leads to having a zero eigenvalue, which leads to having this term. For a general initial condition $f(x)$, and on more general interval $(0, l)$, (4) would become

$$u(x,t) = \frac{1}{l}\int_0^l f(x)dx + \sum_{n=1}^{\infty} a_n e^{-n^2\pi^2 t/l^2}\cos(\frac{n\pi x}{l}) \ .$$

Notice that as $t \to \infty$, all the terms in the summation goes to zero, so $u(x,t) \to (1/l)\int_0^l f(x)dx$, the average of f on the interval. This justifies the informal calculation we did earlier in Section 13, example 3, when originally discussing steady state solutions.

Let us summarize what we have done so far concerning solving for the solutions to the diffusion IBVPs:

1. If you are presented with a problem that has non-homogeneous boundary conditions, transform the problem to one that has homogeneous boundary conditions.

2. Then apply separation of variables method, $u = T(t)\phi(x)$, on the equation and derive a spatial ODE problem (EVP), involving the equation for ϕ **and its boundary conditions**, along with the equation for T. The type of b.c.s on the EVP drives the type of eigenfunction series we have for the solution of the pde problem.

3. Solve the EVP for the set of eigenvalues $\{\lambda_n\}$ and associated eigenfunctions $\{\phi_n\}$.

4. Substitute the eigenvalues into the temporal equation and solve it to obtain the $T_n's$. Then use the (extended) superposition principle to sum all possible contributions to u:

$$u(x,t) = \sum_n A_n T_n(t)\phi_n(x) \ .$$

5. Finally, let $t \to 0$ to obtain $u(x,0) = f(x) = \sum_n A_n \phi_n(x)$, and solve for the Fourier coefficients A_n, using the orthogonality property of the set of eigenfunctions $\{\phi_n\}$. To emphasize, you make use of the initial data as the last process you do!

This strategy is very general, and we will apply it over and over. For example, what if the problem is a wave equation problem?

Example 4:

$$\begin{cases} u_{tt} = c^2 u_{xx} & 0 < x < l\ ,\ t > 0 \\ \\ u(x,0) = f(x) \neq 0 & 0 < x < l \\ \\ u_t(x,0) = g(x) \equiv 0 & \\ \\ u(0,t) = 0 = u(l,t) & t > 0\ . \end{cases}$$

We call this problem the "plucked-string" problem, or "harpsichord problem." Let $u(x,t) = T(t)\phi(x)$ and substitute into the equation:

$$\frac{1}{c^2 T}\frac{d^2T}{dt^2} = \frac{1}{\phi}\frac{d^2\phi}{dx^2} = -\lambda \Rightarrow$$

$$\begin{array}{ll} \frac{d^2\phi}{dx^2} + \lambda\phi = 0 & \quad 0 < x < l \\ \phi(0) = 0,\ \phi(l) = 0 & \end{array}$$

along with

$$\frac{d^2T}{dt^2} + \lambda c^2 T = 0 \quad t > 0\ .$$

From our original example (1), which has the same EVP, we have $\lambda = \lambda_n = (n\pi/l)^2$, $\phi = \phi_n(x) = \sin(\frac{n\pi x}{l})$, $n = 1, 2, 3, \ldots$. For the T equation, substituting $T = e^{rt}$ into the equation gives the characteristic equation $r^2 + \lambda_n c^2 = 0$, so $r = \pm ic\sqrt{\lambda_n}$. Hence, a fundamental set of solutions for the T equation is $\cos(c\sqrt{\lambda_n}t)$, $\sin(c\sqrt{\lambda_n}t)$. Thus

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t)\phi_n(x) = \sum_{n=1}^{\infty}\{a_n\cos(c\sqrt{\lambda_n}t) + b_n\sin(c\sqrt{\lambda_n}t)\}\sin(\frac{n\pi x}{l})$$

To finish off the problem, step 5 in the general procedure, we substitute into the *two* initial conditions to obtain

$$f(x) = \sum_{n=1}^{\infty} a_n \sin(\tfrac{n\pi x}{l})$$

$$g(x) = 0 = c\sum_{n=1}^{\infty} \sqrt{\lambda_n}\, b_n \sin(\tfrac{n\pi x}{l}) \ .$$

A Fourier (sine) series with 0 coefficients represents a Fourier series of the 0 function, and since Fourier series are unique, $b_n = 0$ for all n. If $g(x) = u_t(x,0) \neq 0$, then we would have multiplied both sides by $\sin(\frac{m\pi x}{l})$ and integrated, obtaining an explicit expression for b_n because of the orthogonality property of the eigenfunctions. Instead we do this on the series for f to obtain

$$\int_0^l f(x)\sin(\frac{m\pi x}{l})dx = \sum_{n=1}^{\infty} a_n \int_0^l \sin(\frac{n\pi x}{l})\sin(\frac{m\pi x}{l})dx = \frac{l}{2}a_m \qquad (6)$$

from which we have the formula for a_n for each n, and therefore have the final solution representation

$$u(x,t) = \sum_{n=1}^{\infty} a_n \cos(\frac{cn\pi}{l}t)\sin(\frac{n\pi x}{l}) \ . \qquad (7)$$

Of course, if we are given an explicit function $f(x)$, we would integrate out the left side of (6) (assuming we can do the calculus) and have an explicit set of Fourier coefficients.

Expression (7) for the solution is not very intuitive, but let us make an observation about its structure. Again by the addition formulas, $2\cos(A)\sin(B) = \sin(A+B) - \sin(A-B)$, so

$$\cos(\frac{cn\pi}{l}t)\sin(\frac{n\pi x}{l}) = \frac{1}{2}[\sin(\frac{cn\pi}{l}t + \frac{n\pi x}{l}) - \sin(\frac{cn\pi}{l}t - \frac{n\pi x}{l})]$$

$$= \frac{1}{2}[\sin(\frac{n\pi}{l}(x+ct)) + \sin(\frac{n\pi}{l}(x-ct))]$$

so

$$u(x,t) = \frac{1}{2}\sum_{n=1}^{\infty} a_n \sin(\frac{n\pi}{l}(x+ct)) + \frac{1}{2}\sum_{n=1}^{\infty} a_n \sin(\frac{n\pi}{l}(x-ct)) = F(x+ct) + G(x-ct)$$

So, buried in expression (7) for the solution to example 4 is the general situation discussed initially with regard to the vibrating string equation, namely that the solution is made of left and right-moving waves.

Remark: We are dealing with important, common examples here but they are somewhat special in that they have equations that involve constant coefficients, or have coefficients of a single variable, or coefficients that can be separable. It is not hard coming up with examples of equations that fail to be separable. Two examples are

$$\begin{aligned} &u_{tt} = (t + x + 1)u_{xx} \\ &u_t = u_{xx} + xtu \end{aligned}$$

Summary: What you need to know is captured in the five step procedure on page 8. Do as many problems as time permits to get this separation of variables process down in various problem situations.

Exercises:

1. Modify example 4 to

$$\begin{cases} u_{tt} = c^2 u_{xx} & 0 < x < l \ , \ t > 0 \\ u(x,0) = f(x) \equiv 0 & 0 < x < l \\ u_t(x,0) = g(x) \neq 0 & \\ u(0,t) = 0 = u(l,t) & t > 0 \ . \end{cases}$$

   Call this the "hammered-string" problem (or "piano problem"). Show that the solution has the form $u(x,t) = \sum_{n \geq 1} b_n \sin(c\sqrt{\lambda_n} t) \sin(n\pi x/l)$, and that it can again be written as a sum of left and right moving waves. Show that, for each n, $n = 1, 2, \ldots$,

$$b_n = \frac{2}{c\pi n} \int_0^l g(x) \sin(\frac{n\pi x}{l}) dx \ .$$

   In comparing the coefficients in this problem versus the coefficients in example 4, can you make any physical interpretations about differences between the plucked-string and the hammered-string problems?

2. For the table below verify the eigenvalues and eigenfunctions for various b.c.s given. Then write out what the series is for $u(x,t)$ when the equation is $u_t = Du_{xx}$ and when the equation is $u_{tt} = c^2 u_{xx}$.

left b.c.	right b.c.	eigenvalues	eigenfunctions
$u(0,t)=0$	$u(l,t)=0$	$(\frac{n\pi}{l})^2, n \geq 1$	$\sin(\frac{n\pi x}{l})$
$u(0,t)=0$	$u_x(l,t)=0$	$(\frac{(n-1/2)\pi}{l})^2, n \geq 1$	$\sin(\frac{(n-1/2)\pi x}{l})$
$u_x(0,t)=0$	$u(l,t)=0$	$(\frac{(n-1/2)\pi}{l})^2, n \geq 1$	$\cos(\frac{(n-1/2)\pi x}{l})$
$u_x(0,t)=0$	$u_x(l,t)=0$	$(\frac{n\pi}{l})^2, n \geq 0$	$\cos(\frac{n\pi x}{l})$

3. Consider the problem

$$\begin{cases} u_t = u_{xx} & 0 < x < 1 \ , \ t > 0 \\ u(x,0) = 0 & 0 < x < 1 \\ u(0,t) = 1 \ , \ u(1,t) = 0 & t > 0 \ . \end{cases}$$

Transforming the problem to one with homogeneous boundary conditions, solve the new problem via separation of variables method, then write the solution out for the original variable $u(x,t)$.

4. Consider the problem

$$\begin{cases} u_t = u_{xx} - au & 0 < x < 1 \ , \ t > 0 \ , \ a = \text{constant} \neq 0 \\ u(x,0) = f(x) & 0 < x < 1 \\ u(0,t) = 0 \ , \ u(1,t) = 0 & t > 0 \ . \end{cases}$$

   (a) if $a > 0$, what are the possible steady state solutions?
   (b) solve the time-dependent problem via separation of variables method when $a > 0$. what happens to the solution as $t \to \infty$?
   (c) if $a < 0$, what are the possible steady state solutions?

5. Consider the damped vibrating string problem

$$\begin{cases} u_{tt} + ku_t = c^2 u_{xx} & 0 < x < 1 \ , \ t > 0 \ , \ k, c > 0 \text{ are constants} \\ u(x,0) = f(x) \ , \ u_t(x,0) = 0 & 0 < x < 1 \\ u(0,t) = 0 = u(1,t) & t > 0 \ . \end{cases}$$

Use separation of variables method to find the series solution for the case $k < 2\pi c$. Why is this restriction imposed?

6. Solve the problem

$$\begin{cases} u_t = 0.1 u_{xx} & 0 < x < \pi\ ,\ t > 0 \\ u(x,0) = f(x) & 0 < x < \pi \\ u(0,t) = 0 = u_x(\pi,t) & t > 0\ . \end{cases}$$

7. Solve the problem

$$\begin{cases} u_t = De^{-t}u_{xx} - bu & 0 < x < \pi\ ,\ t > 0\ \ b > 0 \text{ is constant} \\ u(x,0) = R\sin(x) & 0 < x < \pi \quad \text{R is a constant} \\ u(0,t) = 0 = u(\pi,t) & t > 0\ . \end{cases}$$

8. Consider the problem

$$\begin{cases} u_t = Du_{xx} & |x| < L\ ,\ t > 0 \\ u(x,0) = f(x) & |x| < L \\ u(\pm L,t) = 0 & t > 0\ . \end{cases}$$

   Symmetry in an EVP will imply even function solution, i.e. cosines. For this symmetric problem, show that the eigenfunctions are of the form $\cos(\sqrt{\lambda}x)$ and that $u(x,t) = \frac{4}{\pi}\sum_{n\geq 1}\frac{(-1)^n}{2n-1}e^{-(2n-1)^2\pi^2 Dt/L^2}\cos(\frac{(2n-1)\pi x}{2L})$.

9. With $a \in C[0,\infty)$, $a(t) > 0$ for all $t > 0$, consider the problem

$$\begin{cases} u_t = a(t)u_{xx} & 0 < x < 1\ ,\ t > 0 \\ u(x,0) = \sin(\pi x) & 0 < x < 1 \\ u(0,t) = 0 = u(1,t) & t > 0\ . \end{cases}$$

   (a) solve for $u(x,t)$, assuming the general $a(t)$ is known.

(b) here is an **inverse problem:** suppose instead we do not initially know $a(t)$ (except we assume it has the above properties), but we are given an additional piece of information, namely $u(1/2, t) = g(t)$. Here $g(t)$ is positive, continuously differentiable, and $\frac{dg}{dt} < 0$ for $t > 0$, and $g(0) = 1$. Find $a(t)$ in terms of $g(t)$.